TOPIC 2: COMPLEX NUMBERS & TRIGONOMETRIC IDENTITIES

2. Use of complex numbers in deriving trigonometric identities.

To prove the common trigonometric identities, you really only need to memorize two of them. The first one is easy. It's the Pythagorean identity.

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

[*Make sure you know why this one is true!*]

The second one may be a bit strange. Called **Euler's Identity**, it is:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

[*"Euler" is pronounced like "oiler."*]

We are going to begin with a quick review of complex numbers. Then we'll look at a way to derive all the common trigonometric identities so that you will never have to memorize them again.

We shall take the opportunity to review some trigonometry too.

2.1 Review of complex numbers

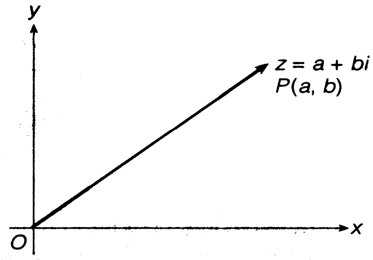
A **complex number** is a number written in the form

$z = x + yi$ where x and y are real numbers and where $i = \sqrt{-1}$.

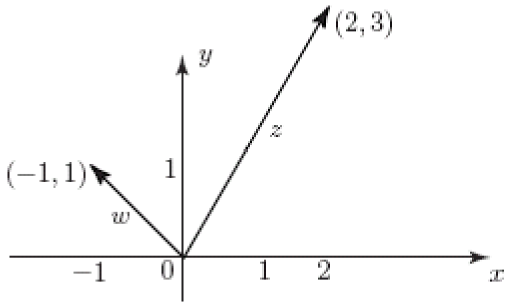
We call x the **real** part of z, and y the **imaginary** part of z:

$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$ e.g.

It can be plotted as a point or a vector in a plane (called the Argand diagram or Argand plane).



<u>Argand diagram</u>	$z = 2+3i$ $\mathrm{Re}(z) =$ $\mathrm{Im}(z) =$ $w =$ $\mathrm{Re}(w) =$ $\mathrm{Im}(w) =$

 $z = 2+3\mathrm{i}$	

Notational convention

If $x = 0$ we write $z = yi$.
If $y = 0$ we write $z = x$.
If $x = y = 0$ we write $z = 0$.

2.2 COMPLEX ALGEBRA

Equality

Two complex numbers $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ are said to be **equal** if and only if $x_1 = x_2$ and $y_1 = y_2$.

Sum and Difference

Complex numbers are added by adding their real parts and adding their imaginary parts.
If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are complex numbers, then their sum is the complex number:

$$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$$

Similarly, the difference of two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is the complex number:

$$z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i$$

Example 1: Add $(7+2i)+(9-5i)$
Solution:

$$(7+2i)+(9-5i) = (7+9)+(2-5)i$$
$$= 16-3i$$

Example 2: Subtract $(4+12i)-(3-15i)$
Solution:

$$(4+12i)-(3-15i) = (4-3)+(12i+15i)$$
$$= 1+27i$$

> *Discussion*:
>
> Ways of writing steps for
> addition,
> subtraction, and
> multiplication
> of complex numbers.

Product

The product of two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is the complex number:

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2)i$$

From $i = \sqrt{-1}$, we get $i^2 = -1$, $i^3 = i^2 i = -1(i) = -i$, $i^4 = i^3 i = -i(i) = -i^2 = 1$

Example 3: Multiply $(1-5i)(-9+2i)$

Solution:

$$\begin{aligned}(1-5i)(-9+2i) &= -9+2i+45i-10i^2\\ &= -9+47i-10(-1)\\ &= 1+47i\end{aligned}$$

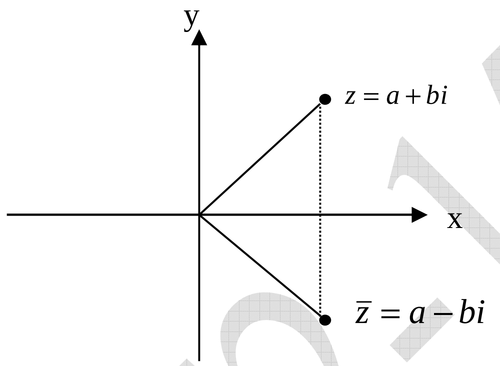
Conjugate $\bar{z}$ or z^*

The **conjugate** of a complex number $z = x + yi$ is the complex number $\bar{z} = x - yi$.

We obtain the $\bar{z}$ by reversing the sign of the imaginary part.

If $z = x + yi$, then $z\bar{z} = (x + yi)(x - yi) = x^2 - xyi + yxi - y^2i^2 = x^2 + y^2$.

If $z = a + bi$ then



Division

To find the quotient of two complex numbers and obtain the result in the form $a + bi$, we multiply both the numerator and the denominator by the complex conjugate of the denominator.

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ with $z_2 \neq 0$.

Then $\dfrac{z_1}{z_2} = \dfrac{x_1 + y_1 i}{x_2 + y_2 i} = \dfrac{(x_1 + y_1 i)(x_2 - y_2 i)}{(x_2 + y_2 i)(x_2 - y_2 i)} = \dfrac{x_1x_2 + y_1y_2}{x_2^{\ 2} + y_2^{\ 2}} + \dfrac{-x_1y_2 + y_1x_2}{x_2^{\ 2} + y_2^{\ 2}}\, i$.

Note: There is no need to memorize this.

Example 4: Express in the form $a + bi$

(a) $\dfrac{3}{9-i}$ (b) $\dfrac{3+4i}{9-2i}$

Solution: (a) $\dfrac{3}{9-i} = \dfrac{3}{9-i}\cdot\dfrac{(9+i)}{(9+i)} = \dfrac{27+3i}{9^2 - i^2} = \dfrac{27+3i}{82} = \dfrac{27}{82} + \dfrac{3}{82}i$

2.3 Review of Trigonometric Functions

(Most of this subsection is extracted from an appendix in Stewart's Calculus.)

The Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

4

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

5

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

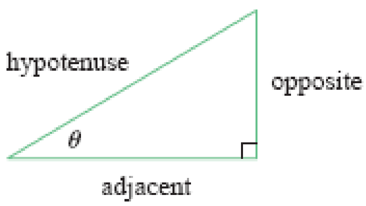


FIGURE 6

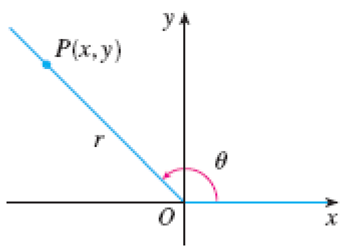


FIGURE 7

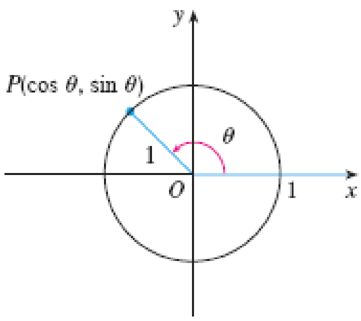


FIGURE 8

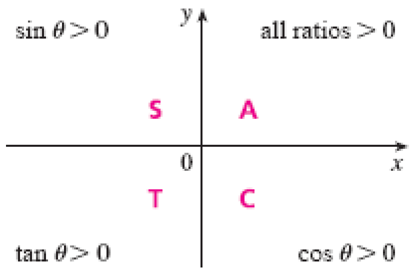


FIGURE 10

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\csc\theta$ and $\cot\theta$ are undefined when $y = 0$. Notice that the definitions in [4] and [5] are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin\theta$ means the sine of the angle whose *radian* measure is θ. For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number, we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write $\sin 3^\circ$ and, with our calculator in degree mode, we find that

$$\sin 3^\circ \approx 0.05234$$

Trigonometric functions of special angles

degrees	0°	30°	45°	60°	90°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–

In class, we shall guide you how to remember this the easy way.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*Und.*	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	*Und*	0

2.4 Euler's Formula and Trigonometric Identities

Euler's Formula says that for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

<u>Angle addition and subtraction identities</u>

Example 1. ***Addition Formulas*** for cosine and sine.

We use the Euler's Formula to express $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ in terms of $\cos\alpha$, $\sin\alpha$, $\cos\beta$ and $\sin\beta$ as follows:

$e^{i\alpha} = \cos\alpha + i\sin\alpha$, $e^{i\beta} = \cos\beta + i\sin\beta$ (Euler's formula)

$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$;
$e^{i(\alpha+\beta)} = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

But $e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$ (Euler's formula again)

$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

Equating the real and imaginary parts on each side of the equation, we have

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

These are the **<u>addition formulas</u>** for the cosine and the sine, respectively.

From these, it would be easy to obtain subtraction formulas and tangent formulas for compound angles. [*See tutorial questions*.]

Euler's formula allows us to rewrite exponentials in terms of trigonometric functions.
It is also useful to be able to go the other way: write trigonometric functions in terms of exponentials.
To derive the necessary formula,
note that, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$,

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

[Do you really know why?]

That is, $e^{i\theta}$ and $e^{-i\theta}$ are conjugates.
By adding or subtracting theses equations, and dividing by 2 or 2*i*, we shall obtain the formulas:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Some examples using these formulas.

Example 2. Consider $\sin^2 x$.

Using $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, we have

$$\sin^2 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 = \frac{-1}{4}\left(e^{i2x} + e^{-i2x} - 2e^0\right) = \frac{-1}{4}(2\cos 2x - 2) = \frac{-1}{2}(\cos 2x - 1)$$

In this example, a power of sine is expressed in a form involving a cosine (not a power of it). This technique will be useful when trying to integrate $\sin^2 x$ later.

Example 3.

Using $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$\sin 3x \cos 4x = \left(\frac{e^{i3x} - e^{-i3x}}{2i}\right)\left(\frac{e^{i4x} + e^{-i4x}}{2}\right)$$

$$= \frac{e^{i7x} - e^{-i7x} - e^{ix} + e^{-ix}}{4i} = \frac{1}{2}\left(\frac{e^{i7x} - e^{-i7x}}{2i} - \frac{e^{ix} - e^{-ix}}{2i}\right)$$

$$= \frac{1}{2}(\sin 7x - \sin x)$$

In this example, the form $\sin mx \cos nx$ (product of a sine and a cosine) is expressed in a form involving the sum of two sines (i.e., a linear combination of two sines).

Similar examples can be done for $\sin mx \sin nx$ and $\cos mx \cos nx$.

See the tutorial questions for more examples.

(nby, Nov 2015)